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WORKING PAPER
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Calculating the Present Value
of Riskless Cash Flows

by
Richard S. Ruback

#1348-83 1483A

Revised
June 1983

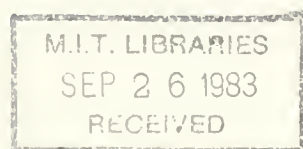
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1. Introduction

This note analyzes a fundamental valuation question:

Is it appropriate for a firm to calculate the present value of riskless after-tax cash flows by discounting at the riskless interest rate or at the after-corporate tax riskless interest rate?

I asked numerous colleagues this question. Each of them thought the answer was obvious. However, approximately half of them thought the obvious answer was the before tax interest rate, whereas the other half believed the answer to be the after-tax interest rate. Introductory finance textbooks also seem to disagree on the answer. While the textbooks do not contain an answer to this specific question, Brealey and Myers (1981) suggest an adjusted present value approach in which the cash flow is valued as the sum of two components: the cash flows discounted at the before-tax interest rate and the realized tax shields discounted at the before-tax interest rate. Weston and Brigham (1981) imply that the cash flows should be discounted at the after-tax riskless rate. Neither Brealey and Myers (1981) nor Weston and Brigham (1981) provide

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I would like to thank Fischer Black, Robert Merton, Wayne Mikkelsen, and Stewart Myers for comments on an earlier draft of this paper.

a rigorous demonstration of why their methods are appropriate.¹

In this note I use arbitrage arguments to prove that the minimal present value of a stream of after-tax riskless cash flows is determined by discounting them at the after-tax discount rate. In the proof the firm constructs the arbitrage by issuing riskless debt with after-tax payments that exactly offset the stream of after-tax cash flows being valued. This equivalent loan is feasible since the firm can secure it with the riskless stream of cash inflows. Furthermore, the equivalent loan is an appropriate arbitrage portfolio in the sense that it eliminates changes in the amount of net debt which would otherwise be associated with the project. Net riskless debt is defined as the present value of riskless cash outflows less the present value of riskless cash inflows. Since the outflows of the equivalent loan exactly offset the riskless inflows from the project in each period, the arbitrage proof does not result in any changes in the firm's riskless net debt.

The arbitrage proof formalizes the standard conceptual definition of the present value of a stream of cash flows: the present value of a stream of riskless cash flows is the minimum amount of money that has to be invested in riskless securities to replicate the stream of cash flows. Since the equivalent loan constructed with riskless debt is feasible, the proceeds from the equivalent loan are, therefore, a lower bound on the present value of the stream of riskless after-tax cash flows. It is a lower bound because the

¹ Disagreement also appears in the literature on leasing. Gordon (1974) uses the before-tax interest rate to value a lease. Henderson (1976) shows that Modigliani and Miller (1963) imply that the after-tax interest rate is appropriate to value a lease. Myers, Dill and Bautista (1976) argue that leasing displaces debt and thus the riskless cash flows from the lease should be discounted at the after-tax interest rate.

arbitrage proof does not ensure that an equivalent loan constructed using debt securities maximizes the proceeds from issuing securities to offset the riskless cash inflows. To the extent that the firm realizes the value of the interest tax shields created by the equivalent loan, the government subsidizes the interest expenses and the opportunity cost of capital for the project is the after-tax riskless interest rate.

The arbitrage proof uses the market price of riskless securities and therefore does not depend on the determinants of the riskless interest rate. The proposition does not imply that there is a corporate advantage to debt financing or that the project should be financed with debt. For example, Miller (1977) presents a model in which there are no personal taxes on income from common stocks and, in equilibrium, the after-corporate-tax interest rate on riskless bonds equals the required rate of return on riskless equity. In the context of the arbitrage proof, Miller's model implies that the present value of the stream of riskless cash flows is unaffected by constructing the equivalent loan using riskless debt or riskless equity.

If there are corporate tax advantages to debt financing, the proceeds from an equivalent loan constructed with riskless debt, and thereby the minimal present value of the stream of cash flows, will be higher than an equivalent loan constructed with riskless equity. Alternatively, if there is a corporate tax disadvantage to debt financing, the after-tax interest rate will exceed the required rate of return on riskless equity.² While the proceeds from an equivalent loan constructed from riskless equities will yield higher proceeds than a loan constructed from debt, the proposition is valid because

² I am not aware of any theory which implies a tax disadvantage to debt financing.

discounting the riskless cash flows at the after-tax riskless interest rate provides a lower bound on the market value of the cash flow. Thus the arbitrage proof, by using the market price of securities, does not depend on the determinates of riskless interest rates.³

2. The present value of riskless cash flows

In this section I show that, when the term structure is uniform and the firm realizes the full value of its interest tax shields, the minimal present value of a riskless after-tax cash flow is determined by discounting it at the after-tax riskless interest rate. Since present values are additive, this proposition implies that the minimal present value of a stream of cash flows is found by discounting them at the after-tax riskless interest rate. When interest rates and marginal tax rates are certain, but not uniform over time, the minimal present value is calculated by discounting the riskless after-tax cash flows at after-tax interest rates that correspond to the term structure and the firm's marginal tax rate. When future one period interest rates are uncertain, I show that the minimal present value of riskless cash flows can be determined using the yields to maturity that are implicit in current pure discount bond prices. Finally, I show that discounting riskless cash flows at the after-tax riskless interest rate is equivalent to the adjusted present value technique recommended by Myers (1974) in which the present value of

³ If firms could issue riskless equity, the arbitrage approach can be used to show that the minimal present value of a riskless cash flow is determined by discounting it at the required rate of return for riskless equity. The required rate of return for riskless equity is not, however, readily observable. The paper focuses on equivalent loans constructed by issuing riskless debt because the riskless interest rate can be easily obtained from the prices of government bonds.

riskless cash flows is determined by discounting the after-tax cash flows at the before-tax riskless interest rate plus the interest tax shields on the equivalent loan discounted at the before-tax riskless interest rate.

Proposition: If a firm realizes the full value of its interest tax shields, the minimal present value of a riskless stream of after tax cash flows is determined by discounting them at the after-tax riskless rate of interest.

Proof: Suppose a firm is going to receive a riskless cash flow of X_T in T periods. The firm pays taxes equal to the corporate tax rate, τ , times the cash flow so that its after-tax cash flow is $X_T(1 - \tau)$. The riskless cash inflow is offset by a riskless cash outflow to prevent a change in the amount of debt outstanding. This is accomplished by constructing an equivalent loan as follows. In the initial period the firm borrows B_0 . In the next period the firm repays $B_0(1 + r)$ to retire its debt. It finances this repayment from an interest tax shield of τB_0 and a new loan of $B_1 = B_0(1 + r(1 - \tau))$. This rollover process continues in each period prior to period T . That is, in period t the firm repays its debt from the prior period, $B_{t-1}(1 + r)$, with its interest tax shield τB_{t-1} and a new loan of $B_t = B_{t-1}(1 + r(1 - \tau))$. In period T the loan from period $T - 1$ is repaid from the after-tax cash flow, $X_T(1 - \tau)$, and the tax shield τB_{T-1} . If the repayment at period T equals the after-tax cash flow, plus the tax shield in period T , the only net cash flow from the riskless cash inflow of $X_T(1 - \tau)$ and the equivalent loan is the initial amount borrowed, B_0 , and this, by definition, is the minimal present value of $X_T(1 - \tau)$.

To find the initial amount borrowed, note that in period T

$$\begin{aligned} X_T(1 - \tau) &= B_{T-1}(1 + r) - \tau B_{T-1} \\ &= B_{T-1}(1 + r(1 - \tau)) . \end{aligned}$$

Since the loan is refinanced each period,

$$B_t = B_{t-1}(1 + r(1 - \tau))$$

and, by recursive substitution,

$$X_T(1 - \tau) = B_0(1 + r(1 - \tau))^T.$$

The present value of $X_T(1 - \tau)$ is B_0 ,

$$PV(X_T(1 - \tau)) = B_0 = \frac{X_T(1 - \tau)}{(1 + r(1 - \tau))^T} . \quad (1)$$

The proposition states that the minimal present value of a single after-tax cash flow is obtained by discounting it at the after-tax riskless interest rate. Since present values are additive, the proposition can be used to value a stream of after-tax cash flows. Denoting $\underline{X}(1 - \tau)$ as the stream of after-tax cash flows,

$$PV(\underline{X}(1 - \tau)) = \sum_{t=1}^T PV(X_t(1 - \tau)) = \sum_{t=1}^T \frac{X_t(1 - \tau)}{1 + r(1 - \tau))^t} \quad (2)$$

The proposition provides the minimal present value of any after-tax riskless cash flows including depreciation tax shields and income subject to capital

gains treatment. To apply the proposition to these cash flows, the numerator in (2), $X_t(1 - \tau)$, is replaced by the after-tax riskless cash flow being valued.

The proposition can be extended to include changing one period interest rates and changes in the marginal corporate tax rate. To incorporate changes in interest rates, the firm uses pure discount bond to replicate the cash flows from the project. The proof requires the firm to realize interest tax shields in each period. If future one period interest rates are certain, then the proof of the proposition is not affected substantially. The equivalent loan is constructed by rolling over one period discount bonds. That is, in period $t-1$ the firm borrows

$$B_{t-1} = B_0 \prod_{j=1}^{t-1} (1 + r_j(1 - \tau))$$

and repays the loan in period t with the tax shield of $\tau_t B_{t-1}$ and a new loan of $B_t = B_{t-1}(1 + r_t(1 - \tau))$ where r_t is the one period interest rate in time t . Similarly, if future tax rates are certain, the equivalent loan is constructed using the interest tax shields that will be realized: the loan of B_{t-1} is repaid with tax shields of $\tau_t r_t B_{t-1}$ and a new loan of $B_t = (1 + r_t(1 - \tau_t))$ where τ_t is the marginal corporate tax rate in time t . Thus when future one period interest rates and tax shields are certain, the proceeds from the equivalent loan, which equals the minimal present value of a certain cash flow received in T periods, is:

$$PV(X_T) = B_0 = \frac{X_t}{\prod_{t=1}^T (1 + r_t(1 - \tau_t))} \quad (3)$$

If future one period interest rates are uncertain, the firm can use pure discount bonds to value the project by short selling discount bonds to offset

the after-tax cash flows from the project. The proceeds from these short sales is the minimal net present value of the project. The tax code allows the firm to deduct the yield on the implicit loan balance in each period. That is, in period t the firm is allowed to deduct interest of $B_0 R_T (1 + R_T)^{t-1}$ where B_0 is the initial amount borrowed and R_T is the yield on the pure discount bond that matures in period T .

The proceeds of the equivalent loan are determined by calculating the amount of pure discount bonds that offsets all of the cash inflows and tax shields. Suppose, as in the proposition, that the firm has a riskless project with a riskless after-tax cash flow of X_T in T periods and that the firm realizes the full value of interest tax shields. Define Z_T as the amount of pure discount bonds that are sold short to offset the cash flow in period T and R_T as the yield on a T period riskless pure discount bond. In period T the firm receives the riskless after-tax cashflow from the project $X_T(1 - \tau)$ and the interest tax shield from the pure discount bond that matures in period T . The firm must pay $Z_T(1 + R_T)^T$ to cover its short sale. Z_T is determined by equating the inflows and outflows in period T :

$$Z_T(1 + R_T)^T = X_T(1 - \tau) + \tau Z_T R_T (1 + R_T)^{T-1}.$$

Rearrangement yields,

$$Z_T = \frac{X_T(1 - \tau)}{(1 + R_T)^{T-1}(1 + R_T(1 - \tau))}$$

The pure discount bond that matures in period T creates tax shields in each period. The firm offsets these intermediate tax shields by short selling pure discount bonds that mature in the period in which the tax shield is realized. For example, in period $T-1$ the firm receives an interest tax shield of $\tau Z_T R_T (1 + R_T)^{T-2}$ from the pure discount bonds that mature in period

T and an interest tax shield of $\tau Z_{T-1} R_{T-1} (1 + R_{T-1})^{T-2}$ from the bonds that mature at T-1. The amount of bonds short sold to offset the cash flow in period T-1, Z_{T-1} , is determined by equating inflows and outflows:

$$Z_{T-1} (1 + R_{T-1})^{T-1} = \tau Z_T R_T (1 + R_T)^{T-2} + \tau Z_{T-1} R_{T-1} (1 + R_{T-1})^{T-2}$$

Rearrangement yields:

$$Z_{T-1} = \frac{\tau Z_T R_T (1 + R_T)^{T-2}}{(1 + R_{T-1})^{T-2} (1 + R_{T-1} (1 - \tau))} \quad (4)$$

The bonds that mature in periods T and T-1 provide tax shields in period T-2 which are offset using a discount bond that matures in T-2. The amount of bonds short sold to offset the period T-2 tax shields are:

$$Z_{T-2} = \frac{\tau Z_T R_T (1 + R_T)^{T-3} + \tau Z_{T-1} R_{T-1} (1 + R_{T-1})^{T-3}}{(1 + R_{T-2})^{T-3} (1 + R_{T-2} (1 - \tau))} \quad (5)$$

The general expression for the amount of pure discount bonds that are short sold to offset the tax shields that are realized in period T-K is:

$$Z_{T-K} = \frac{\tau \sum_{t=0}^{K-1} Z_{T-t} R_{T-t} (1 + R_{T-t})^{T-t-1}}{(1 + R_{T-K})^{T-K-1} (1 + R_{T-K} (1 - \tau))} \quad \text{for } K = 1, 2, \dots, T-1 \quad (6)$$

The equivalent loan constructed by borrowing Z_t dollars of pure discount bonds that mature in time t is an appropriate arbitrage portfolio. It is feasible since the firm can secure the loans with the riskless cash flow from the project and the riskless tax shields. Also, while future one period interest rates are uncertain, the current prices of pure discount bonds and their associated yields to maturity are known. Finally, the equivalent loan

offsets all cash flows associated with the project so that there is no change in the amount of net riskless debt.

The proceeds from the equivalent loan created by the short sales is

$$PV(X_T(1 - \tau)) = \sum_{t=1}^T Z_t \quad (7)$$

and equals the minimal net present value of the project. Note that in the absence of a uniform and certain term structure of interest rates the present value of a riskless after-tax cash flow is not determined by discounting at the after-tax yield to maturity. This occurs because the intermediate tax shields from the equivalent loan are discounted at yields relevant to the periods in which the tax shields are realized. Expression (7) does, however, reduce the (1) if the term structure is uniform.⁴

⁴ When the term structure is uniform, the yield on discount bonds of any maturity equals the one period rate, r . Equation (3) becomes:

$$Z_T = \frac{X_T(1 - \tau)}{(1 + r)^{T-1}(1 + r(1 - \tau))}$$

The general expression for Z_{T-K} reduces to:

$$Z_{T-K} = \frac{\tau \sum_{t=0}^{K-1} Z_{T-K}}{1 + r(1 - \tau)}$$

The present value of the cash flow equals the sum of the Z_t . Note that

$$Z_T + Z_{T-1} = Z_T \left[\frac{1 + r}{1 + r(1 - \tau)} \right]$$

Similarly,

$$Z_T + Z_{T-1} + Z_{T-2} = \frac{1 + r}{1 + r(1 - \tau)} (Z_T + Z_{T-1}) = Z_T \left[\frac{1 + r}{1 + r(1 - \tau)} \right]^2$$

This recursive substitution implies that

$$\sum_{t=1}^T Z_t = Z_T \left[\frac{1 + r}{1 + r(1 - \tau)} \right]^{T-1}$$

Substituting the expression for Z_T yields

$$\sum_{t=1}^T Z_t = \frac{X_T(1 - \tau)}{(1 + r)^{T-1}(1 + r(1 - \tau))} \left[\frac{1 + r}{1 + r(1 - \tau)} \right]^{T-1} = \frac{X_T(1 - \tau)}{(1 + r(1 - \tau))^T}$$

The proceeds from the equivalent loan can also be expressed as:

$$PV(X_T(1-\tau)) = \frac{X_T(1-\tau)}{(1+R_T)^T} + \tau \sum_{t=1}^T \frac{\sum_{K=J}^T Z_K R_K (1+R_K)^{t-1}}{(1+R_t)^t} \quad (8)$$

Thus, the minimal present value of a riskless after-tax cash flow can be calculated as the sum of the cash flow discounted at the before-tax yield plus the interest tax shields in each period discounted at the corresponding before-tax yield. This formulation is, therefore, similar to the adjusted present value approach recommended by Myers (1974).

Expression (8) shows that the tax shields from the equivalent loan depend on the value of the riskless after-tax cash flow. For example, when the term structure is uniform, equation (8) simplifies to:

$$PV(X_T(1-\tau)) = V_0^T = \frac{X_T(1-\tau)}{(1+r)^T} + \sum_{t=1}^T \frac{\tau V_{t-1}^T}{(1+r)^t}$$

where V_t^T is the minimal present value of $X_T(1-\tau)$ in period t .

According to the proposition proven earlier, the value of the riskless cash flow grows at the after-tax discount rate,

$$V_t^T = V_{t-1}^T (1 + r(1-\tau)).$$

Thus, while the minimal present value of a riskless after-tax cash flow can be expressed as the after-tax cash flow discounted at the before-tax interest rate plus the interest tax shields discounted at the before-tax rate, the minimal present value of the cash flow is required to determine the balance of the equivalent loan and the associated interest tax shields. The minimal present value of the after-tax cash flow can be easily computed using the after-tax discount rate. Pragmatically, determining the present value of a riskless after-tax cash flow by discounting it at the after-tax interest rate

appears to be much easier than the adjusted present value approach because the present value of the cash flow (including the present value of the associated interest tax shields) determines the appropriate tax shields to include in the adjusted present value formulation.

Conclusions

This paper uses arbitrage arguments to prove that the minimal market value of a stream of riskless after-tax cash flows is determined by discounting them at the after-tax riskless interest rate. In the proof, the firm constructs an equivalent loan such that the loan payments exactly offset the stream of riskless after-tax cash flows. The proceeds from this equivalent loan equal the minimal present value of the riskless after-tax cash flows. Since the firm realizes interest tax shields from this equivalent loan, the appropriate discount rate to value the stream of riskless after-tax cash flows is the after-tax riskless rate.

It is worthwhile to examine the argument for using the before-tax interest rate to value riskless after-tax cash flows in light of the proof that the after-tax discount rate is appropriate. Arguments for using before-tax discount rate typically focus on the investors' opportunity cost of capital instead of the corporation's cost of capital. For example, investors can invest in riskless bonds directly. The proceeds from this investment, before personal taxes, is the riskless interest rate. Thus the investors' opportunity cost of capital for riskless investments is the before-corporate tax riskless rate. The after-tax earnings on riskless corporate investment must, therefore, equal or exceed the before-tax riskless rate to compensate investors for their opportunity cost of capital.

The argument for using the before-tax interest rate to value riskless after-tax cash flows ignores the corporation's ability to generate interest

tax shields by borrowing against the riskless cash flows. Since this borrowing offsets the riskless after-tax cash flow, the appropriate discount rate for riskless after-tax cash flows is the corporation's cost of capital for riskless borrowing which is the after-tax riskless interest rate.

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